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Derivations on Novikov Algebras

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Novikov algebras are nonassociative algebras introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. As one of the most important topics in the study of Novikov algebras, the derivation is related to many fields such as the vector fields, the automorphisms, the cohomology theory, and so on. In this paper, we study the derivations and inner derivations of Novikov algebras. We also give their classification in low dimensions.

KEY WORDS: Novikov algebras.

1. INTRODUCTION

Hamiltonian operators have close relation with certain algebraic structures (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983, 1984; Gel'fand and Diki, 1975, 1976; Gel'fand and Dorfman, 1979; Xu, 1995). Gel'fand and Diki introduced formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983). Gel'fand and Dorfman (1979) found more connections between Hamiltonian operators and certain algebraic structures. Dubrovin, Balanskii, and Novikov studied similar Poisson structures from another point of view (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983, 1984). One of the algebraic structures appearing in Gel'fand and Dorfman (1979) and Balinskii and Novikov (1985) which is called a "Novikov algebra" by Osborn (Osborn,

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1992a,b, 1994; Xu, 1996, 1997), was introduced in connection with the Poisson brackets of hydrodynamic type

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

A Novikov algebra A is a vector space over a field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x, y, z) = (y, x, z) \tag{1.2}$$

and

$$(xy)z = (xz)y, \tag{1.3}$$

for $x, y, z \in A$, where

$$(x, y, z) = (xy)z - x(yz). \tag{1.4}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy Eq. (1.2). Left-symmetric algebras are nonassociative algebras arising from the study of affine manifolds, affine structures, and convex homogeneous cones (Bai and Meng, 2000; Burde, 1998; Kim, 1986; Vinberg, 1963).

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx, \tag{1.5}$$

defines a (subadjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote the left and right multiplication respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. If every R_x is nilpotent, then A is called right-nilpotent or transitive. The transitivity corresponds to the completeness of the affine manifolds in geometry (Kim, 1986; Vinberg, 1963).

There has been some progress in the study of Novikov algebras, such as the fundamental structure theory of a finite-dimensional Novikov algebra over an algebraically closed field with characteristic 0 (Zel'manov, 1987), the infinite-dimensional simple Novikov algebras (Osborn, 1992b, 1994), the finite-dimensional simple Novikov algebras over an algebraically closed field with prime characteristic (Xu, 1996), the Poisson structures on Novikov algebras (Xu, 1997), the classification of Novikov algebras over the complex number field in low dimensions (Bai and Meng, 2001a), the realization of Novikov algebras (Bai and Meng, 2001b,c), the invariant bilinear forms on Novikov algebras (Bai and Meng, 2001d,e), and so on. However, because of the nonassociativity, there are still many open questions which are quite different with any known algebras.

Among them, one of the most important topics is the derivation. The derivation plays an important role not only in algebra itself, but also in many related fields.

For example, in geometry, it is related to vector fields (Knopp, 1988). It is also well known that the derivation algebra of an algebra is a Lie algebra and its automorphism group is a Lie group whose Lie algebra is just its derivation algebra (Sagle and Walde, 1973). This means that the derivations can be regarded as a kind of "linearization" of the automorphisms which play a key role in the classification of algebras. Moreover, like in the case of Lie algebras, we can define a cohomology theory (see details in Burde, 1998 and Bai and Meng, 2001b) and the derivations are is just the 1-cocycles, that is, the derivation algebra equals to $Z^1(A, A)$. Hence, the derivations and the derivation algebras have many applications.

In this paper, we discuss the derivations of Novikov algebras. The paper is organized as follows. In Section 2, we briefly give some basic properties of derivations. We also discuss how to obtain some derivations on a lot of Novikov algebras based on a kind of realization theory of Novikov algebras. In Section 3, we discuss the inner derivations of Novikov algebras. In Section 4, we give the classification of derivation and inner derivation algebras over **C** in dimension 3. In Section 5, we give some discussion for the results in the previous sections.

2. THE DERIVATIONS OF NOVIKOV ALGEBRAS

Let A be a Novikov algebra. Let End(A) denote the set of all linear transformations of A. Then End(A) is a vector space. Furthermore, End(A) is a Lie algebra with respect to the bracket

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_2\mathcal{A}_1, \qquad \forall \mathcal{A}_1, \quad \mathcal{A}_2 \in \mathrm{End}(A). \tag{2.1}$$

A derivation D of A is a linear transformation $D \in \mathrm{End}(A)$ satisfying

$$D(xy) = (Dx)y + x(Dy), \qquad \forall x, \quad y \in A. \tag{2.2}$$

This equation can be rewritten in terms of right and left multiplications: $D \in \mathrm{End}(A)$ is a derivation if and only if any of the following equations is satisfied:

$$[D, L_x] = L_{Dx}, \qquad \forall x \in A, \tag{2.3}$$

$$[D, R_y] = R_{Dy}, \qquad \forall y \in A. \tag{2.4}$$

It is well known that the set $D(A)$ of all derivations of A is a Lie subalgebra of End(A). Recall an automorphism ϕ of A is an invertible linear transformation satisfying

$$\phi(xy) = \phi(x)\phi(y), \qquad \forall x, \quad y \in A \tag{2.5}$$

Then $\exp tD$ is an automorphism for any $D \in D(A)$ and $t \in \mathbf{R}$, and $D(A)$ is the Lie algebra of the automorphism group Aut(A) which is the set of automorphisms (Bai and Meng, 2001e).

The derivations of Novikov algebras have many properties which belong to all nonassociative algebras. For example, we have the so-called Jordan-Chevalley decomposition: there exists unique $D_s, D_n \in D(A)$ satisfying the conditions: $D = D_s + D_n$, D_s is semisimple, D_n is nilpotent, D_s and D_n commute.

Besides these common properties, there is one property that is related to the subadjacent Lie algebra: let $\mathrm{Der}(A)$ be the Lie derivation algebra of the subadjacent Lie algebra of A. Then by Eq. (1.5), we have $D(A) \subset \mathrm{Der}(A)$.

In general, the derivation algebra $D(A)$ is very complex, although it is a Lie subalgebra of $\mathrm{Der}(A)$. And it is also difficult to obtain the nonzero derivations. However, on the basis of the realization theory of Novikov algebras in Bai and Meng (2001b,c), we can obtain some derivations on a lot of Novikov algebras. Next we discuss them in details, followed by a brief introduction of self-contained algebras.

Let A be a commutative associative algebra with the product $(,\cdot,)$ and D be its derivation. Then the new product

$$x *_a y = x \cdot Dy + a \cdot x \cdot y, \tag{2.6}$$

makes $(A, *_a)$ become a Novikov algebra for $a = 0$ by Gel'fand (1979), for $a \in \mathbf{F}$ by Filipov (1989) and for a fixed element $a \in A$ by Xu (1997). We show that the algebra $(A, *) = (A, *_0)$ given by Gel'fand is transitive, and the other two kind of Novikov algebras given by Filipov and Xu are the special deformations of the former (Bai and Meng, 2001b). Moreover, a deformation theory of Novikov algebras (Bai and Meng, 2001b,c) is constructed and we prove that the Novikov algebras in dimension ≤ 3 can be realized as the algebras defined by Gel'fand and their compatible infinitesimal deformations. We conjecture that this conclusion can extend to higher dimensions. In particular, in dimension 2 and 3, a lot of transitive Novikov algebras (except (A6) with $l = 0$, (A8), (A10)) and almost nontransitive Novikov algebras (except only (E1)) can be realized through Eq. (2.6).

Claim. Let D' be a derivation of $(A, \cdot)$. If $D'D = D'D$, then D' is a derivation of $(A, *_a)$ for $a = 0$ and $a \in \mathbf{F}$. In particular, in this case, $D \in D(A, *_a)$. If $D'D = D'D$ and $D'a = 0$, then D' is a derivation of $(A, *_a)$ for $a \in A$.

In fact, we have

$$\begin{aligned} D'(x *_a y) &= D'(x \cdot Dy + a \cdot x \cdot y) \\ &= D'(x \cdot Dy + x \cdot D'Dy + D'a \cdot x \cdot y + a \cdot D'x \cdot y + a \cdot x \cdot D'y \\ &= D'x *_a y + x *_a D'y + x \cdot (D'D - DD')y + D'a \cdot x \cdot y. \end{aligned}$$

Here, for $a = 0$ and $a \in \mathbf{F}$, we let $Da = 0$. Hence, the above claim holds. $\square$

Example 2.1. Let us give the derivation algebras of two-dimenesional Novikov algebras over the complex number field which the classification is given in Bai and

Meng (2001a). Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^{n} c_{11}^{k} e_k & \cdots & \sum_{k=1}^{n} c_{1n}^{k} e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} c_{n1}^{k} e_k & \cdots & \sum_{k=1}^{n} c_{nn}^{k} e_k \end{pmatrix}, \tag{2.7}$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^{n} c_{ij}^{k} e_k$. Moreover, under the same basis, any derivation D of A can be determined by a matrix, that is,

$$D = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \tag{2.8}$$

For any two-dimenisonal noncommutative Novikov algebras, we have known that it can be realized through Eq. (2.6) (Bai and Meng, 2001b,c). Thus, we have

Characteristic matrix	Derivation algebra	Associated $(A, \cdot)$	Remark
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$gl(2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$		Commutative
(T2) $\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} \\ 0 & 2a_{11} \end{pmatrix}$		Commutative
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	0		Commutative
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$		Commutative
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$		Commutative
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = 0$	D(T3) = D (N3)
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix}$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_2$	
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}$	(N3) with $D = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_1 + e_2$	

Characteristic matrix	Derivation algebra	Associated $(A, \cdot)$	Remark
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$	(N3) with $D = \begin{pmatrix} l-1 & 0 \\ 0 & 0 \end{pmatrix}$ and $a = e_2$	D(N6) = D(N3)

3. THE INNER DERIVATIONS

The most important subset of the derivation algebra of a Novikov algebra is the set of inner derivations. A general theory for inner derivations is given in Schafer (1949).

Let A be a Novikov algebra. The Lie subalgebra $\mathcal{L}(A)$ generated by all linear transformations $L_x, R_y (\forall x, y \in A)$ is called the Lie multiplication algebra (or Lie transformation algebra). Let $\mathcal{M} = R(A) + L(A)$ denote the set spanned by all L_x, R_y. Set

$$\mathcal{M}_1 = \mathcal{M}, \qquad \mathcal{M}_i = [\mathcal{M}_1, \mathcal{M}_{i-1}]. \tag{3.1}$$

Then

$$\mathcal{L}(A) = \mathcal{M}_1 + \cdots + \mathcal{M}_i + \cdots \tag{3.2}$$

It is the smallest Lie algebra containing $\mathcal{M} = R(A) + L(A)$.

A derivation D of A is called an inner derivation if $D \in \mathcal{L}(A)$. From Eqs. (2.3) and (2.4), it is easy to see that the set $\mathrm{Inn}(A)$ of all inner derivations is a (Lie) ideal of the Lie algebra $D(A)$. The inner derivation corresponds the inner automorphism of A. $\mathrm{Inn}(A)$ also may be regarded as a candidate for the space of 1-coboundaries $B^1(A, A)$. If so, the cohomology $H_1(A, A)$ is just $D(A)/\mathrm{Inn}A$.

Claim. The Lie transformation algebra of a Novikov algebra A is

$$\mathcal{L}(A) = L(A) + \mathbf{F}\left[R_{e_1}, R_{e_2}, \ldots, R_{e_n}\right], \tag{3.3}$$

where $e_1, \ldots, e_n$ is a basis of A and $\mathbf{F}[R_{e_1}, R_{e_2}, \ldots, R_{e_n}]$ denote the polynomial algebra generated by $R_{e_1}, \ldots, R_{e_n}$.

In fact, any element of $R(A) + L(A)$ has the form $R_x + L_y$. By Eqs. (1.2) and (1.3), we have

$$[L_x, L_y] = L_{[x,y]}, [L_x, R_y] = R_{xy} - R_y R_x, [R_x R_y] = 0$$

Therefore

$$\left[L_{x_1} + R_{y_1}, L_{x_2} + R_{y_2}\right] = L_{[x_1,y_1]} + L_{[x_1 y_2]} - R_{y_2} R_{x_1} - R_{x_2 y_1} + R_{y_1} R_{x_2}.$$

Moreover, we have

$$\begin{aligned}\left[R_{x_1}, R_{x_2}, L_{x_3}\right] &= R_{x_1}\left[R_{x_2}, L_{x_3}\right] + \left[R_{x_1}, L_{x_3}\right]R_{x_2} \\ &= R_{x_1}\left(R_{x_2}R_{x_3} - R_{x_2x_3}\right) + \left(R_{x_1}R_{x_3} - R_{x_1x_3}\right)R_{x_2} \\ &= 2R_{x_1}R_{x_2}R_{x_3} - R_{x_1}R_{x_2x_3} - R_{x_1x_3}R_{x_2};\end{aligned}$$

and for any n,

$$\begin{aligned}&\left[R_{x_1}R_{x_2}\cdots R_{x_n}, L_{x_{n+1}}\right] \\ &\quad = R_{x_1}\cdots R_{x_{n-1}}\left[R_{x_n}, L_{x_{n+1}}\right] + \left[R_{x_1}\cdots R_{x_{n-1}}, L_{x_{n+1}}\right]R_{x_n} \\ &\quad = R_{x_1}\cdots R_{x_{n-1}}\left(R_{x_n}, R_{x_{n+1}} - R_{x_n,x_{n+1}}\right) + \left[R_{x_1}\cdots R_{x_{n-1}}, L_{x_{n+1}}\right]R_{x_n}.\end{aligned}$$

Hence, by induction on n and Eq. (3.1) and (3.2), we have that any element in the Lie transformation algebra $\mathcal{L}(A)$ has the form

$$L_{x_1} + R_{x_1^1} + R_{x_1^2}R_{x_2^2} + R_{x_1^3}R_{x_2^3}R_{x_3^3} + \cdots \tag{3.4}$$

By the commutativity of R_{e_i}, we have $\mathcal{L}(A) = L(A) + \mathbf{F}[R_{e_1}, R_{e_2}, \ldots, R_{e_n}]$. $\square$

Since every right multiplication of a transitive Novikov algebra is nilpotent, we have

Corollary *Let A be a transitive Novikov algebra. Then there exists N such that $\mathbf{F}[R_{e_1}, R_{e_2}, \ldots, R_{e_n}]$ in Eq. (3.3) becomes the polynomial algebra in degree less than N.*

It is not easy to give the explicit formula for the inner derivations. However for some special cases, we can easily obtain some inner derivations. By Eqs. (1.2) and (1.3), we can have

(a) $L_x \in D(A)$ *if and only if* $(ax)b = 0, \forall a, b \in A$.
(b) $R_x \in D(A)$ *if and only if* $(ab)x - a(bx) = (ax)b, \forall a, b \in A$.
(c) $\mathrm{ad}x = L_x - R_x \in D(A)$ *if and only if* $a(bx) = (ab)x, \forall a, b \in A$.
(d) *Let* $T(A) = \{x \in A \mid \mathrm{ad}x \in D(A)\}$. *Then* $T(A)$ *is an associative subalgebra of* A.
(e) $\mathrm{ad}_L(A) = \{\mathrm{ad}x \mid \mathrm{ad}x \in D(A)\}$ *is a* (Lie) *ideal of* $\mathrm{Der}(A)$ *if and only if*

$$D(T(A)) \subseteq T(A), \qquad \forall D \in \mathrm{Der}(A)$$

(f) $\mathrm{Int}(A) = \mathrm{ad}(A) = \{\mathrm{ad}x \mid x \in A\}$ *is an ideal of* $D(A)$ *if and only if* A *is associative.*
(g) *If* $D(A) = \mathrm{Der}(A)$, *then* A *is associative.*
(h) *If* A *has an identity, then for every* $x \in A$, $x \neq 0$, *neither* L_x *nor* R_x *is a derivation of* A.
(i) *(i) If all derivations of the subadjacent Lie algebra are inner (that is,* $\mathrm{Der}(A) = \{\mathrm{ad}x \mid x \in A\}$*), then* $D(A) = \mathrm{Inn}(A)$.

Example 3.1. Since the (Lie) derivations of 2-dimensional non-Abelian Lie algebra are inner, all derivations of (T3), (N4), (N5), and (N6) are inner. For other cases, we have

$$\mathrm{Inn(T1)} = 0; \mathrm{Inn(T2)} = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}; \mathrm{Inn(N1)} = \mathrm{Inn(N2)} = \mathrm{Inn(N3)} = 0.$$

4. THE DERIVATION ALGEBRA OF THREE-DIMENSIONAL NOVIKOV ALGEBRAS

In this section, we give the derivation algebras and the inner derivation algebras of three-dimensional Novikov algebras over the complex number field **C** (the classification is given in Bai and Meng (2001a)). At first, we give the (Lie) derivation algebras of three-dimensional Lie algebras over **C** (the classification is given in Jacobson (1962) as follows:

(1) A is Abelian: $\mathrm{Der}(A) = gl(3) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$;

(2) $A = \langle e_1, e_2, e_3 \mid [e_3, e_2]$

$$= e_1, [e_3, e_1] = [e_2, e_1] = 0\rangle : \mathrm{Der}(A) = \begin{pmatrix} a_{22}+a_{33} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

(3) $A = \langle e_1, e_2, e_3 \mid [e_3, e_2]$

$$= e_2, [e_3, e_1] = [e_2, e_1] = 0\rangle : \mathrm{Der}(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix};$$

(4) $A = \langle e_1, e_2, e_3 \mid [e_3, e_1]$

$$= e_1, [e_3, e_2] = e_2, [e_2, e_1] = 0\rangle : \mathrm{Der}(A) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix};$$

$A = \langle e_1, e_2, e_3 \mid [e_3, e_1] = e_1, [e_3, e_2]$

$$= le_2, [e_2, e_1] = 0, |l| \leq 1, l \neq 0, 1\rangle : \mathrm{Der}(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix};$$

(5) $A = \langle e_1, e_2, e_3 \mid [e_3, e_1]$

$$= e_1, [e_3, e_2] = e_1 + e_2, [e_2, e_1] = 0\rangle : \mathrm{Der}(A) = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix};$$

We give our main results in the following table:

Characteristic matrix	Derivation algebra	Inner derivation algebra	Remark
(A1) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	0	D(A1) = Der(A1)
(A2) $\begin{pmatrix}0&0&0\\0&0&0\\0&0&e_2\end{pmatrix}$	$\begin{pmatrix}2a_{33}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\a_{31}&0&0\end{pmatrix}$	
(A3) $\begin{pmatrix}0&0&0\\0&e_1&0\\0&0&e_1\end{pmatrix}$	$\begin{pmatrix}2a_{22}&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&-a_{23}&a_{22}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\a_{21}&0&0\\a_{31}&0&0\end{pmatrix}$	
(A4) $\begin{pmatrix}0&0&0\\0&0&e_1\\0&e_1&e_2\end{pmatrix}$	$\begin{pmatrix}3a_{33}&0&0\\2a_{32}&2a_{33}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\a_{31}&0&0\end{pmatrix}$	
(A5) $\begin{pmatrix}0&0&0\\0&0&e_1\\0&-e_1&0\end{pmatrix}$	$\begin{pmatrix}a_{22}+a_{23}&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\a_{21}&0&0\\a_{31}&0&0\end{pmatrix}$	D(A5) = Der(A5)
(A6) $\begin{pmatrix}0&0&0\\0&e_1&e_1\\0&-e_1&le_1\end{pmatrix}$	$\begin{pmatrix}2a_{22}&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&-la_{23}&a_{22}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\a_{21}&0&0\\a_{31}&0&0\end{pmatrix}$	
(A7) $\begin{pmatrix}0&0&0\\0&0&e_1\\0&le_1&e_2\end{pmatrix}$ $l \neq 1$	$\begin{pmatrix}3a_{33}&0&0\\2a_{32}l&2a_{33}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\2a_{32l}&0&0\\a_{31}&a_{32}&0\end{pmatrix}$	
(A8) $\begin{pmatrix}0&0&0\\0&0&0\\0&e_1&e_2\end{pmatrix}$	$\begin{pmatrix}3a_{33}&0&0\\a_{32}&2a_{33}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\a_{32}&0&0\\a_{31}&a_{32}&0\end{pmatrix}$	
(A9) $\begin{pmatrix}0&0&0\\0&0&0\\0&e_2&0\end{pmatrix}$	$\begin{pmatrix}a_{11}&0&0\\0&a_{22}&0\\a_{31}&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&a_{22}&0\\0&0&0\end{pmatrix}$	
(A10) $\begin{pmatrix}0&0&0\\0&0&0\\0&e_2&e_1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&a_{22}&0\\a_{31}&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&a_{22}&0\\a_{31}&0&0\end{pmatrix}$	D(A10) = Inn(A10)

Characteristic matrix	Derivation algebra	Inner derivation algebra	Remark
(A11) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}$ $\lvert l\rvert \leq 1, l \neq 0$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l = 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l = 1$	
	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l \neq 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & la_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l \neq 1$	
(A12) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_2 + e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{11} & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(A13) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}a_2 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \frac{1}{2}a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \frac{1}{2}a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	D(A13) = Inn(A13)
$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	0	0	
(B1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(B2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(B3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	D(B3) = Inn(B3)
(B4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ 0 & 0 & e_1 + e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	D(B4) = Inn(B4)
(B5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_2 & 0 \\ le_1 & 0 & e_1 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	D(B5) = Inn(B5)

Characteristic matrix	Derivation algebra	Inner derivation algebra	Remark
(C1) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(C2) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(C3) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	
(C4) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ 0 & 0 & e_1+e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	
(C5) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(C6) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	
(C7) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3+e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	
(C8) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$	D(C8) = Der(C8)
(C9) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3 \end{pmatrix}$ $l \neq 1, 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} l-1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	
(C10) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & 0 & e_3+e_2 \end{pmatrix}$ $l \neq 1$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $l = 0$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $l = 0$	

Characteristic matrix	Derivation algebra	Inner derivation algebra	Remark
	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$ $l \neq 0$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$ $l \neq 0$	
(C11) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(C12) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(C13) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & ke_2 & e_3 \end{pmatrix}$ $l, k \neq 1, 0$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l = k$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l = k$	
	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l \neq k$	$a_{11}\begin{pmatrix} l-1 & 0 & 0 \\ 0 & k-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $l \neq k$	
(C14) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_1+e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(C15) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & e_1+le_2 & e_3 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$a_{21}\begin{pmatrix} l-1 & 0 & 0 \\ 1 & l-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(C16) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ -a_{11} & a_{11} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	
(C17) $\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3+e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{21} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$	

Characteristic matrix	Derivation algebra	Inner derivation algebra	Remark
(C18) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ a_{31} & a_{31} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{31} & 0 \end{pmatrix}$	
(C19) $\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3+e_1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{31} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{31} & 0 \end{pmatrix}$	D(C19) = Inn(C19)
(D1) $\begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 2a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	
(D2) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 2a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	
(D3) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1+e_2 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	D(D3) = Inn(D3)
(D4) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 2a_{11} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 2a_{11} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	D(D4) = Inn(D4)
(D5) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{1}{2}e_1 & 0 & e_3+e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	D(D5) = Inn(D5)
(D6) $\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ le_1 & (2l-1) & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 2a_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 2a_{11} & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$	D(D6) = Inn(D6)
$l = \frac{1}{2}, 1$ (E1) $\begin{pmatrix} 0 & 0 & 0 \\ -e_1 & 0 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	D(E1) = Inn(E1)

5. DISCUSSION AND CONCLUSION

From the results given in the previous sections, we can obtain:

1. In dimensions 2 and 3, the derivation algebra of any Novikov algebra which is not semisimple (the direct sum of fields) is nonzero.
2. In dimensions 2 and 3, the inner derivation algebra of a noncommutative algebra is nonzero; in particular, the inner derivation algebra of a nonassociative Novikov algebra is nonzero. And $\dim \mathrm{Inn}(A) \leq \dim A$. Furthermore, when A is a Novikov algebra in dimensions 2 or 3 and A is neither type (N4) nor type (C8), we have $\dim \mathrm{Inn}(A) < \dim A$.
3. In dimensions 2 and 3, $D(A) = \mathrm{Inn}(A) \neq 0$ if and only if A is isomorphic to one of the following types: (T3), (N4), (N5), (N6), (A10), (A13), (B3), (B4), (B5), (C19), (D3), (D4), (D5), (D6), (E1).

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